Long Time Scale Molecular Dynamics Using Least Action.

B. A. Gladwin [*] T. Huber [†]

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1 Theory

Olender and Elber [1] reformulate the least action principle in terms of the error between an approximate path and the physical path. In t his section the we outline the key elements of this approach and highlight the details of our algorithm.

Any path defined by the boundary conditions $[\mathbf{x}(t_{initial}), \mathbf{x}(t_{final})]$ can be approximated by the set of coordinates $\mathbf{x}_{approx}(t)$. The probability of a transition between a pair of these coordinates $\mathbf{x}_{approx}(t_i), \mathbf{x}_{approx}(t_j)$ can be expressed as the conditional probability in the following form.

$$P\left(\mathbf{x}_{approx}(t_j)|\mathbf{x}_{approx}(t_i); \Delta t\right), \quad \Delta t = t_j - t_i$$

The transition which maximises this probability is the nearest transtion to the true trajectory. By applying this reasoning to each transition along the trajectory we can located the most likely path.

If we assume that the error which arises from the discrete step is gaussian around the trtue trajectory then it is possible to approximate the error correlation between two steps as

$$\langle \epsilon(t_i)\epsilon(t_j)\rangle \approx \frac{\sigma^2 \delta_{ij}}{\Delta t} \tag{1}$$

[*]Department of Mathematics, University of Queensland, Brisbane, Australia. mailto:gladwin@maths.uq.edu.au

[†]Department of Mathematics, University of Queensland, Brisbane, Australia

This allows us to define the conditional probability of a particular transition as

$$P\left(\mathbf{x}(t_i)|\mathbf{x}(t_i+\Delta t);\Delta t\right)=\left(\frac{1}{2\pi\langle\epsilon^2\rangle}\right)^{d/2}\exp\left[-\frac{\left(\mathbf{x}(t_i+\Delta t)-\mathbf{x}_{approx}(t_i+\Delta t)\right)^2}{2\langle\epsilon^2\rangle}\right] \tag{2}$$

This is turn allows us to formulate an expression for the conditional probability of an entire trajectory as the product of all of the transition probabilities of that trajectory.

$$\Pi_i P\left(\mathbf{x}(t_i)|\mathbf{x}(t_i+\Delta t);\Delta t\right)\approx\exp\left[-\left(\frac{1}{2\sigma^2}\right)\int\left(\epsilon(t)\right)^2 dt\right] \tag{3}$$

In this way we have a single exponential function representing the probability that a certain trajectory is correct expressed in terms of the error from the true path. Intuitively, the path with the highest probability has zero error at each point in the path.

Onsager and Machlup state that the probability that a given trajectory $\mathbf{x}(t)$ will satisify Langevin's equation of motion is given by the equation

$$P\left[\mathbf{x}(t)\right]\propto\exp\left[\frac{-\mathcal{S}}{k_BT}\right] \tag{4}$$

where the k_B is Boltzmann's constant and T in this expression is equivalent to the temperature of the system. Comparing this expression to equation 3 we get an expression for the action given by

$$\mathcal{S}=\int\left(\epsilon(t)\right)^2 dt \tag{5}$$

This equation is a version of the Onsager Machlup Action.

It is now necessary to formulate a function for the error in the path. As in Olender and Elber's paper we have chosen to formulate this error based on Newtons equations of motion.

For the correct trajectory, we assume that the following result holds at every point along the path.

$$\mathbf{F_i}-m_i\mathbf{a_i}=0$$

An approximation to this path can be rewritten as

$$\frac{dV(\mathbf{x}_{approx}(t),t)}{d\mathbf{x}}-\mathbf{m}\frac{d^2\mathbf{x}_{approx}}{dt^2}(t)=\epsilon(t) \tag{6}$$

Where the $\epsilon(t)$ is now the error in the path. By using this expression in equation 5, the action can be represented

$$\mathcal{S} = \int \left(\mathbf{F}(t) - \mathbf{m}\ddot{\mathbf{x}}(t)\right)^2 dt \tag{7}$$

Using this formualtion of the action instead of the one proposed by Hamilton **??** forces the action to be positive and non-zero for all trajectories which do not follow the exact path. Furthermore the exact trajectory will have a value for the action of zero.

In this way we have reformulated the path search as a global minimisation problem. It remains now to specify an approximate path in terms of a parameter set and apply standard minimisation algorithms.

In this analysis the expansion chosen is the fourier sine series. The motivation, advantages and disadvantages with this choice are described in the following paragraph.

Specifying the path as

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{a}t + \sum_i b_i \sin\left(\frac{i\pi t}{\tau}\right) \tag{8}$$

where $\mathbf{x}_0$ is the initial positions of the particles and $\mathbf{a}$ represents an initial linear approximation to the path. This expansion is clearly useful in situation where the positions return to initial configurations as in the case of molecular motors. There is alsoan ablity to amplify or damp physical properties, such as viscous or drag effect and high frequencey oscillations by adding or removing terms in the expansion. This type of expansion also lends itself to the use of heiraical path searching regimes. Successive minimisation runs with increasing numbers of expansions allows efficient minimisation by first optimising the gross movements of the molecule using lower modes and the adjusting the path using the highr modes. For the expansion shown in equation 8 there is a key limitation when applied to situations which have initial forces. In these cases there will always be a non-zero final action arising from the second derivative of 8 being initially zero for all experiments. This causes equation 7 to be non-zero initially. It was necessary to cater for this in experiment two to achieve the desired zero-action path and is seen clearly in experiment three. This does not show a deficiency with the algorithm, but with the choice of expansion and this limitation should be kept in mind when selecting test cases for the process.

The recent work of passerone and parrinello [7] uses an iterative process to attain dynamical trajectories from know boundary conditions. Their approach avoids the use of second derivatives by successive minimisation steps which progressivley optomise the energy and minise the aciton. In this paper, we have choosen to use the analytic second derivative to enable the implementation of fast optomising regimes such as the conjugate gradient technique and avoid multiple minimisation runs. In the following paragraphs, some of the key derivations and the basic algorthm used in the results section are outlined.

Since the motion of the a particular particle is governed by its interactions with surrounding particles a potential energy function which captures these interactions needs to be built. In the following results sections the potential energy functions are drawn from literature and are expressed in terms of only the coordinates of the particles. Knowing these potential energy functions we can find the derivative of the action as

$$\begin{aligned} \mathcal{S} &= \Sigma \left(\mathbf{F}(t) - \mathbf{m}\ddot{\mathbf{x}}(t)\right)^2 \\ \frac{\partial \mathcal{S}}{\partial \mathbf{b}} &= 2\left(\mathbf{F}(t) - \mathbf{m}\ddot{\mathbf{x}}(t)\right)\left(\frac{\partial \mathbf{F}(t)}{\partial \mathbf{b}} - \mathbf{m}\frac{\partial \ddot{\mathbf{x}}(t)}{\partial \mathbf{b}}\right) \end{aligned}$$

where

$$\mathbf{F}(t) = -\frac{d\mathcal{V}(t)}{d\mathbf{x}}$$

and the $\ddot{\mathbf{x}}(t)$ represents the time derivative of the path given by the expansion in equation 8. By calculating these derivatives analytically it is possible to employ standard minimisation algorithms such as conjugate gradient minimisation in order to locate the set of coefficients $\{\mathbf{b}\}$ for which the action is zero. This algorithm is summariesed below

1. Define a initial guess of the path in terms of and adjustable set of parameters $\{\mathbf{b}\}$.

2. Specify all the interactions of intrest in terms of a potential energy function $\mathcal{V}(t)$.

3. Evaluate the action and its derivatives at each time step from the expressions 7 and 1 and apply conjugate gradient minimisation in order to locate the coefficient which minimises the action.

There are a number of advantages and disadvantages in formulating the problem in this least action framework. These shall now be discussed in greater detail.

One of the more apparent advantage to this algorithm is greatly adjustable to suit the computational limitations. In theory, the coarseness of

the time samplling used in the simulation does not effect the existance of a stable trajectory. This allows the location in time of key events to discovered at a coarse grained level and then investigated at greater detail. In practice this is not the case. An insufficient number of samlple points would potentially avoid key events in the transition, resulting in a path which does not capture the real dynamics accurately. This results from the algorithm just searchingfor a path whose action is zero. While there may be more than one of these according to initial conditions, there will always be at least one stable trajectory.

The number of coefficents can also be adjusted to suit available computaton time. As mentioned previously, the fourier expansion allows the user to treat the optomisation part of the algorithm hierarchically. This greatly improves the efficiency of the algorithm by dealing with the low modes of oscillation first and building the fine adjustments from this. The adjustability of the number of expansions is both a strength and a weakness. If the user undestimates the number of expansions necessary to bring the system to the final conditions in the given time then an additional force is needed to satisify the boundary conditions. This force is not catered for in this technique and will show in the action calculation. It is therefore necessary progressively increase the number of expansions till no change is seen in the action calculation.

A computational advantage to this formulation is it's suitability to parallel processing. As there is no need to calculate the path in the order it occours in, processors can be assigned individual time slices and all path calculation can take place simultaneously.

Finally, as this is constructed in the form of a boundary value problem, the transition process will always finish at the final state. This is not a guarentee in a finite time for traditional forward integrating regime.

These advantages make this a useful, efficient technique for solving problems where the boudary information is known.

These difficulties are being investigated at present and other expansions are being tested for applicability.

2 Results

In the following section we will overview some of the results achieved using the method outlined in the previous sections along with the discussion concerning the effectiveness of the technique and difficulties encountered.

The first of the systems will be a simple two bodied harmonic oscillator.

This is a particularly simple system given the expansion that we have chosen in this overview. The next two systems we investigate here are using a standard Lennard-Jones interaction potential. The first of these is a straight forward three bodied rotation, the accuracy and efficiency of the method applied to this system will be investigate. Finally some initial testing on a seven particle Lennard-Jones cluster as tested by Dellago *et. al.* [5].

Convergence to a stable final path is very rapid in the first two of these systems. The final system converges slower, however applying the iterative scheme outlined in the theory section, convergence to the final solution is rapid.

2.1 Harmonic Potentials.

The harmonic potential used in experiment one is given by the function in equation 9 and operates across all bonds.

$$\mathcal{V}(t) = \frac{1}{2} k_b \left((x_j - x_i) - x_{equlib} \right)^2 \tag{9}$$

Where the equlibrium separation (x_{equlib}) and the spring constant (k_b) are set to 0 and $\frac{1}{2}$ respectively. The mass for both particles is equal and the initial separation between the particles is zero.

2.1.1 Experiment 1: Two Particle System.

The time scale for the simulation is chosen so that the particles return to their starting configuration. In this one space, one time dimensional problem the particles have been allow to crossover.

For the purposes of the algorithm the inital choice of coefficients for the minimisation is random. The figure below shows a plot of path given by the initial guess in dashed line and the final solution for the system is shown by the solid lines. The figure on the right is a demonstration of the convergence to the final solution in terms of the number of iterations.

It is clear that in this simple experiment the intuitive path is reached in very few iterations. The overall action for this final path is zero. In this simple experiment, this approach is very efficient. In the next two experiments we investigate gradually more complex situations.

2.2 Lennard-Jones Potentials.

For the following two experiments, the interaction potential is the non-bonded Lennard-Jones potential. This is shown in the following equation

This interaction exists across all pairs of atoms.

In the following experiments the initial positions of the particles has been chosen as the force equlibrium positions. This is important for achieving zero action pathways in this type of simulation.

In the following two experiments, three particles are rotating in a clockwise direction. The initial and final separations for all pairs of particles in each of the systems is the same. This means that the initial and final forces are the same. For this to be the case it was necessary to balance the angular momentum of the particles with the lennard-jones force. The purpose of placing the particles at these positions was to ensure that the true path would be circular. This has been done for comparison purposes so that the simple expansion we have used would be able to closely approximate the path.

Because of the simplicity of the present expansion choice it has been necessary to choose cases where the expansion will not damage the experimental results. Further work will certainly improve the expansion choice and allow broader application of this method. In the following two experiments the potential function used is the Lennard-Jones potential for simulating the ..... This is expressed mathematically as

and operates across all pairs of atoms. In both experiments all the particles masses are equal and the coefficients A_{ij} and B_{ij} are chosen so that the particles are initially at their equlibrium positions.

2.2.1 Experiment 2: Three Particle System

The figure below shows the setup for experiment two. Again all particle are chosen to have equal mass and the transition time (T) is chosen arbitrarily.

The rotation in this experiment requires an initial force. This initial force will show up in the action calculation yielding a non-zero final action. In this simple example, it is possible to adjust the starting position of the particles in order to achieve a zero final action. In this case the resulting path will be circular and the particles will stay at their equlibrium positions for the whole transition.

The right hand plot in the figure below shows the difference betweenn the initial guess at the coefficients and the final paths of the particles. The dashed line show the initial choice of the coefficients and the solid line show the final path. The result shown uses 21 coefficients. As in the previous experiment, the figure on the right shows the convergence to this final solution. From this second plot it is again clear that the final result is converged to this solution within a few iterations.

The following figures were done using just one expansion as an approximation to the path since the additional expansions used in the figure !!!!!!!!!ref!!!!!!!!! do not significantly alter the path. The following figure compares the force on one of the atoms on the right, and the difference between the final path found using the one expansion and the true circular path.

This demonstraights the need for a more accurate expansion by showing that the inability of the expansion to accuratly represent the true path is the main limitation in achieveing a dynamical solution.

2.2.2 Experiment 3: Seven Particle System

This final experiment has used the same potential as in the previous experiment. The three rotating particles have been embedded in a symmetrical seven bodied cluster. The figure below shows the experimental setup. In this case the four new particles do not rotate.

As a first approximation to the path we have chosen to use the result achieved form the previous experiment. The figure below shows a plot of the solution which was converged to.

The convergence to the final solution in this experiment was much slower than in previous examples. However, by applying the heirical optimisation resume the minimisation can be done in a short time. In this case the final path did not yield a zero action path. The final result of the minimisation was a local minima in the action surface. This means that the final path achieved was the nearest to the true path possible using this particular expansion.

3 Discussion and Conclusions

It is clear from this initial analysis that while the choice of expansion is not sufficient for general cases the algorithm is sufficient for

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References

[1] R. Olender and R. Elber, Calculation of classical trajectories with a very large time step: Formalism and numerical examples. *Journal of Chemical Physics* **105** (1996), 9299-9315.

[2] P. Ferrara, J. Apostolakis and A. Caflisch, Targeted molecular dynamics simulations of protein unfolding. *Journal of Physical Chemistry B* **104** (2000), 4511-4518.

[3] P. Ferrara, J. Apostolakis and A. Caflisch, Computer simulations of protein folding by targeted molecular dynamics. *Proteins-Structure Function and Genetics* **39** (2000), 252-260.

[4] C. Dellago, P.G. Bolhuis, F.S. Csajka and D. Chandler, Transition path sampling and the calculation of rate constants. *Journal of Chemical Physics* **108** (1998), 1964-1977.

[5] C. Dellago, P.G. Bolhuis and D. Chandler, Efficient transition path sampling: Application to Lennard- Jones cluster rearrangements. *Journal of Chemical Physics* **108** (1998), 9236-9245.

[6] C. Dellago, P.G. Bolhuis and D. Chandler, On the calculation of reaction rate constants in the transition path ensemble. *Journal of Chemical Physics* **110** (1999), 6617-6625.

[7] D. Passerone, M. Ceccarelli and M. Parrinello, A concerted variational strategy for investigating rare events. *Journal of Chemical Physics* **118** (2003), 2025-2032.

[8] D. Passerone and M. Parrinello, Action-derived molecular dynamics in the study of rare events. *Physical Review Letters* **8710** (2001), art. no.-108302.

[9] M. Parrinello, Action-derived molecular dynamics in the study of rare events. *Abstracts of Papers of the American Chemical Society* **221** (2001), 140-PHYS.